Periodical waves, domain walls, and modulational instability in dispersive quadratic nonlinear media

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We show that the three-wave mixing interaction in media with quadratic nonlinearity, and dispersion up to the second order, possess classes of mutually sustained periodical nonlinear traveling waves and domain walls (or kink pairs). We also predict a particular kind of Benjamin-Feir, or modulational instability. This mechanism may also affect the stability of certain classes of localized waves. These phenomena have importance in different branches of physics whenever parametric interactions may occur through *quadratic* nonlinearities.

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The concept of nonlinear localized waves and solitons is ubiquitous in physics. Nonlinear optics offers unique opportunities to investigate the propagation of solitons in cubic media [i.e., described by nonlinear Schrödinger-like equations (NLSEs) [1,2]], sine-Gordon solitons of self-induced trasparency [3], and solitons of the dispersive three-wave mixing (TWM) interaction [4–6]. In particular, the TWM equations are widespread as model equations of mixing phenomena due to *quadratic* nonlinearities in different branches of physics such as plasmas, water, or acoustic waves [6]. Although we specifically refer to the case of degenerate TWM [7] between a first and second harmonic in optics, the concepts which follow may be extended to nondegenerate processes also in other physical contexts. TWM equations are integrable only in the presence of first-order dispersion (i.e., group-velocity mismatch) [4–7]. Although, one may suspect that such integrability may be lost when dispersion at second order is accounted for, TWM admits the existence of localized bright and dark solitary wave forms [8–11]. These localized waves are the object of renewed interest in view of their potential application exploiting their nonlinear phases in all-optical switching [12], and at power levels strongly reduced with respect to localized waves of cubic media (described, e.g., by nonintegrable NLSEs [2]). In this paper we show that TWM possesses a particular class of nonlinear waves in the presence of both first- and second-order dispersion. They are periodic traveling waves (bright and dark waves are particular cases) and domain walls between the nonlinear eigenmodes of the process. As we will show, these waves are also relevant in the context of field theory. We further show that second-order dispersion is responsible for a mechanism of instability, which involves the generation of two sideband pairs, and is reminiscent of the Benjamin-Feir, or modulational instability (MI), known so far only for *cubic* nonlinearities [13,14]. MI may also lead to the spontaneous breaking of dark and kink solitary pairs into organized patterns.

In a transparent dispersive medium with quadratic nonlinearity the slowly varying field envelopes $A_{1,2} = A_{1,2}(z,t)$, at frequencies of ω_0, and $2\omega_0$, respectively, obey the coupled partial differential equations [9–11]

$$
\begin{aligned}
i\frac{\partial A_1}{\partial z} + i\frac{\beta'}{2}\frac{\partial A_1}{\partial t} + \frac{\beta_1''}{2}\frac{\partial^2 A_1}{\partial t^2} + \chi A_2 A_1^* e^{i\Delta k z} &= 0, \\
i\frac{\partial A_2}{\partial z} - i\frac{\beta'}{2}\frac{\partial A_2}{\partial t} + \frac{\beta_2''}{2}\frac{\partial^2 A_2}{\partial t^2} + \chi A_1^2 e^{-i\Delta k z} &= 0,
\end{aligned}
\tag{1}
$$

where $\Delta k = k(2\omega_0) - 2k(\omega_0)$ is the wave vector mismatch, $\beta' \equiv k'(2\omega_0) - k'(\omega_0) = dk/d\omega|_{2\omega_0} - dk/d\omega|_{\omega_0}$ is the group-velocity mismatch (i.e., first-order dispersion), $\beta_{1,2}'' = d^2k_{1,2}/d\omega^2|_{\omega_0,2\omega_0}$ is the second-order dispersion, χ is the nonlinearity coefficient, and the time t is measured in a frame moving at velocity $[k'(2\omega_0)+k'(\omega_0)]/2$. Introducing the new dimensionless variables $\xi = z|\beta_1''|/t_0^2 \equiv z/z_d$, $\sigma = t/t_0$ (z_d is the dispersion length associated to the time scale t_0), $u_1 = \sqrt{2}\chi z_d A_1$, $u_2 = \chi z_d A_2$, Eqs. (1) may be casted in Hamiltonian form as

$$
\frac{\partial u_j}{\partial \xi} = i\frac{\delta H}{\delta u_j^*} \quad (j=1,2)\ , \tag{2}
$$

where $H = \int_{-\infty}^{+\infty} \mathcal{H} d\sigma$ and $\mathcal{H}$ is the Hamiltonian density

$$
\begin{aligned}
\mathcal{H} &= \mathcal{H}(u_j, u_j^*, u_{j,\sigma}, u_{j,\sigma}^*) \\
&= \frac{u_1^2 u_2^* e^{-i\delta k\xi} + (u_1^2)^* u_2 e^{i\delta k \xi}}{2} \\
&\quad + \sum_{j=1,2}\left[(-1)^j \frac{\delta}{4}(iu_{j,\sigma}u_j^* - iu_{j,\sigma}^* u_j) + \frac{\beta_j |u_{j,\sigma}|^2}{2}\right]\ ,
\end{aligned}
\tag{3}
$$

where $\delta k \equiv \Delta k z_d$, $\beta_1 \equiv \mathrm{sgn}(\beta_1'')$, $\beta_2 \equiv \beta_2''/\beta_1''$, and $\delta \equiv \beta' z_d/t_0$. Equations (1)–(3) also govern the interaction of cw beams focused in one transverse dimension (i.e., σ) with $\beta_j'' = 1/2k_j$, and β' due to birefringence walk off.

To obtain the traveling (i.e., periodic or localized) waves of Eqs. (2) and (3) we seek for solutions in the usual form

$$
u_1(\xi,\sigma) = x(\tau)e^{i\phi_1(\xi,\sigma)}\ ,\quad u_2(\xi,\sigma) = y(\tau)e^{i\phi_2(\xi,\sigma)}, \tag{4}
$$

where x and y are real functions of $\tau \equiv \sigma - v\xi$ and $\phi_1 \equiv \mu_1\xi + \nu\sigma$, $\phi_2 \equiv \mu_2\xi + 2\nu\sigma$ depends linearly on space and time. We obtain the two coupled equations

$$\ddot{x} = \frac{2}{\beta_1}\left[\theta_1 x - xy\right], \quad \ddot{y} = \frac{2}{\beta_2}\left[\theta_2 y - \frac{x^2}{2}\right], \quad (5)$$

where the dot stands for $\partial/\partial\tau$, and where we set $\nu = \delta/(2\beta_2 - \beta_1)$, $v = \delta(2\beta_2 + \beta_1)/[2(2\beta_2 - \beta_1)]$ in order to cancel first-order dispersive terms, and $2\mu_1 - \mu_2 = \delta k$ which permits us to eliminate the dependence on ξ. $\theta_1 \equiv \mu_1 + \delta\nu/2 + \beta_1\nu^2/2$ and $\theta_2 \equiv \mu_2 - \delta\nu + 2\beta_2\nu^2$ represent second-order corrections of (μ_1, μ_2) due to the frequency shift $(\nu, 2\nu)$. We emphasize that Eq. (5) and its solutions have a wide interest, beyond TWM, in field theory. In fact, invariant traveling-wave solutions to coupled fields which evolve according to either complex Schrödinger and real Klein-Gordon equations, or real and complex Klein-Gordon equations obey Eq. (5) with a proper redefinition of the parameters [15]. Without any loss of generality we renormalize Eq. (5) in order to deal with a single parameter. Henceforth we consider only anomalous dispersions (i.e., $\beta_1 = 1$, $\beta_2 > 0$) since the extension to the normal dispersion regime follows from the trivial invariant transformation $(\beta_{1,2}, \theta_{1,2}, x, y) \to (-\beta_{1,2}, -\theta_{1,2}, -x, -y)$ of Eq. (5). After rescaling Eq. (5) as

$$x \to x|\theta_2|/\sqrt{\beta_2}, \quad y \to y|\theta_2|/\beta_2, \quad \tau \to \tau\sqrt{\beta_2/|\theta_2|}\ , \quad (6)$$

we cast Eq. (5) in the canonical Hamiltonian form $\dot{x} = \partial H_2/\partial p_x$, $\dot{p_x} = -\partial H_2/\partial x$ (and identical with $x \to y$), where the momentum $(p_x, p_y) \equiv (\dot{x}, \dot{y})$, and H_2 reads as

$$H_2 = H_2(x, y, p_x, p_y) = \frac{p_x^2 + p_y^2}{2} + V(x, y). \quad (7)$$

Here $V(x, y) = -\theta x^2 - \eta y^2 + x^2 y$, where $\eta \equiv \theta_2/|\theta_2| = \pm 1$, is a nonintegrable potential of the Hénon-Heilis family [16] which depends on the single parameter $\theta \equiv \theta_1\beta_2/(\beta_1|\theta_2|)$. Hence, though the Hamiltonian (7) may exhibit chaotic trajectories, a class of bounded periodical evolutions on a torus and separatrices correspond to periodical and solitary waves of Eq. (1), respectively. A subset of these solutions may be obtained explicitly by considering straight line trajectories (i.e., $y = cx$). In this case, the "force" $(\ddot{x}, \ddot{y})$ must be parallel to the trajectory (x, cx). This requires $c = 1/\sqrt{2}$ (also $c \to -c$, but then $x \to -x$) and $\theta = \eta = \pm 1$. In this case Eq. (7) reduces to the integrable one degree of freedom Hamiltonian

$$H_1 = H_1(x, p_x) = \frac{3}{4}p_x^2 - \frac{3}{2}\eta x^2 + \frac{1}{\sqrt{2}}x^3. \quad (8)$$

We may express the periodical solutions of Eq. (8) in terms of Jacobian elliptic functions by inverting the quadrature integral. We obtain the class of symbiotic snoidal waves (following the terminology introduced by Korteweg–DeVries [18]) of Eqs. (2) and (3),

$$x(\tau) = b_0 + b_1 sn^2(\alpha\tau|k)\ , \quad y(\tau) = x(\tau)/\sqrt{2}, \quad (9)$$

where $b_0 = x_1$, $b_1 = x_3 - x_1$, $\alpha^2 = (x_1 - x_2)/(3\sqrt{2})$, and $k = \sqrt{(x_1 - x_3)/(x_1 - x_2)}$ is the modulus of the elliptic sine sn , and $x_n (n = 1, 2, 3)$ are the roots of the cubic equation $E - V(x) = E + (3\eta/2)x^2 - x^3/\sqrt{2} = 0$, obtained from Eq. (8) with $E \equiv H_1(x, p_x)$. In terms of the angle $\psi = \tan^{-1}\{[2\sqrt{-(\eta + E)E}]/(2E + \eta)\}$, these roots read as

$$x_n = \frac{\eta}{\sqrt{2}} + \sqrt{2}\cos\left[\frac{\psi}{3} + \frac{2\pi}{3}(n-1)\right] (n = 1, 2, 3)\ , \quad (10)$$

where $0 \le \psi \le \pi$ and $x_2 \le x_3 \le x_1$. We obtain $b_0 = \eta/\sqrt{2} + \sqrt{2}\cos(\psi/3)$, $b_1 = -\sqrt{6}\sin(\psi/3 + 2\pi/3)$, $\alpha^2 = \sin(\psi/3 + \pi/3)/\sqrt{3}$, and $k^2 = [\sin(\psi/3 + 2\pi/3)]/[\sin(\psi/3 + \pi/3)]$. The snoidal pair (9) has period $2K(k)/\alpha$, where $K(k)$ is the elliptic integral of the first kind. For $k \to 1$ (i.e., $\psi \to 0$) $K(k) \to \infty$, and Eq. (9) reduces to the solitary wave solution

$$x(\tau) = \frac{1}{\sqrt{2}}\left[2 + \eta - 3\tanh^2\left(\frac{\tau}{\sqrt{2}}\right)\right],$$
$$y(\tau) = \frac{x(\tau)}{\sqrt{2}}. \quad (11)$$

For $\eta = \theta = -1$ this requires $E \to -1$ and Eq. (11) represents a self-trapped dark pair. Conversely, for $\eta = \theta = 1$, $k \to 1$ in the limit $E \to 0$, and Eq. (11) reduces to a bright pair of the form $x(\tau) = (3/\sqrt{2})\cosh^{-2}(\tau/\sqrt{2})$. Both these dark and bright pairs are twin waves which travel with a common locked group velocity in the presence of a nonvanishing walk off δ. In this respect they generalize the stationary solutions previously reported for $\delta = 0$ in optics [8–11] and in the context of field theory for the bright case [15]. In the phase space associated to the Hamiltonian (7) the bright solution is a homoclinic separatrix trajectory which connects the origin at $\tau = -\infty$ and $\tau = +\infty$, whereas the dark pair is a separatrix which emanates from the equilibrium solution $(x, y) = (-\sqrt{2}, -1)$. In general, the potential $V(x, y)$ has three equilibrium points: the origin and the points $(x, y) \equiv (\bar{u}_1, \bar{u}_2) = (\pm\sqrt{2\eta\theta}, \theta)$, which represent phase-locked eigensolutions of the cw [i.e., $\partial/\partial t = \partial^2/\partial t^2 = 0$ in Eq. (1)] two-wave interaction (arising from bifurcations of the $u_1 = 0$ eigenmode [17]). Therefore a heteroclinic separatrix connecting the points $(-\sqrt{2\eta\theta}, \theta)$, and $(\sqrt{2\eta\theta}, \theta)$, which are equipotential, represents a new localized solution in the form of a symbiotic domain wall (or a kink pair). Bounded solutions of this kind may exist only for $\eta = -1$ and $\theta < 0$. In this case the points $(\overline{u}_1, \overline{u}_2)$ are saddle points of $V(x, y)$. These trajectories are no longer lines in the plane (x, y) and must be found from the nonintegrable two-dimensional Hamiltonian (7). We searched for such solutions by using a standard numerical shooting method. We integrated Eq. (5) with initial conditions sitting on the unstable manifold of one saddle, using θ as a shooting parameter in order to seek trajectories that tend asympotically to the other saddle (we then propagate backward in order to verify the solution). We found a family of countable exotic solutions which may be ordered in terms of increasing complexity. In Fig.1 we show the trajectories in the plane (x, y), superimposed to equipotential curves, as well as the wave forms of two low-order domain wall solutions obtained for $\theta = -0.556$ and $\theta = -0.311$, respectively. The first-$[x(\tau)]$ and second-harmonic $[y(\tau)]$ fields are always *an-*

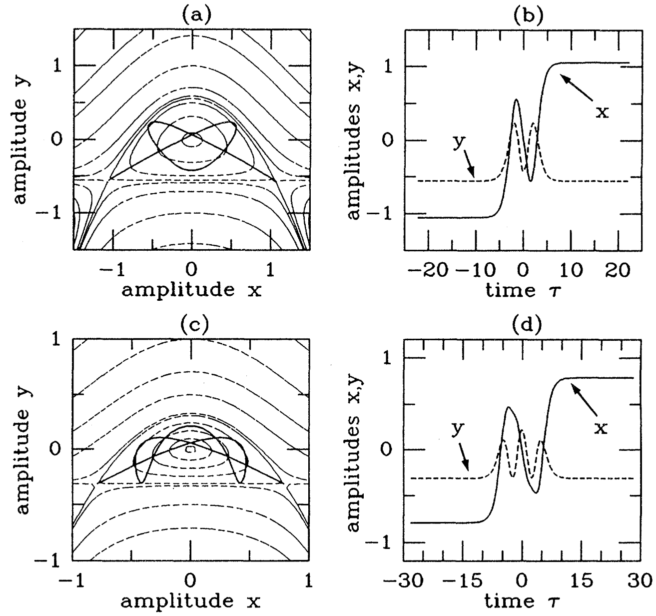

FIG. 1. The localized domain wall of the two-wave interaction for $\eta = -1$. Heteroclinic trajectory superimposed to the contour plot of the potential [(a) and (c)] and corresponding field amplitudes $x = x(\tau)$, and $y = y(\tau)$ vs τ [(b) and (d)]. The curves correspond to $\theta = -0.556$ [(a) and (b)] and $\theta = -0.311$ [(c) and (d)].

tisymmetric and *symmetric*, respectively. It is worth noting that these domain wall are reminiscent of those of birefringent cubic media, where the two field components (frequency degenerate but with different polarization) obey coupled NLSEs [19].

We checked numerically the stability of the periodical waves (9) by integrating a finite-difference scheme of Eqs. (2) and (3) with initial conditions slightly perturbed from the exact solution. The observed bounded oscillatory behavior along the propagation direction suggests that the snoidal (as well as bright) waves are stable in a wide range of ψ values. Conversely, the interplay of dispersion and nonlinearity may lead to instabilities of the eigenmodes $(\bar{u}_1, \bar{u}_2)$ causing the formation of temporal (or transverse) structures. In cubic media, MI (or a Benjamin-Feir instability [13]) is a ubiquitous mechanism. In nonlinear optics it entails the build up of a sideband pair by virtue of four-photon processes involving the decay of two pump photons into down- and upshifted photon pairs [14]. We show here that MI occurs also in quadratic nonlinear media, involving three-photon processes. A photon at frequency $2\omega_0$ annihilates, creating a photon twin $(\omega_0 - \Omega, \omega_0 + \Omega)$, whereas photons at the sideband frequency $2\omega_0 \pm \Omega$ are created through the TWM process $\omega_0 + (\omega_0 \pm \Omega) \to 2\omega_0 \pm \Omega$. The unstable detunings Ω may be obtained by inserting into Eqs. (2) and (3) the ansatz

$$u_1(\xi,\tau) = \left[\bar{u}_1 + \epsilon_{11} e^{i\Omega\tau} + \epsilon_{12} e^{-i\Omega\tau}\right] e^{i\theta\xi},$$
$$u_2(\xi,\tau) = \left[\bar{u}_2 + \epsilon_{21} e^{i\Omega\tau} + \epsilon_{22} e^{-i\Omega\tau}\right] e^{i(2\theta-\delta k)\xi}, \quad (12)$$

where $\vec{\epsilon} = \epsilon(\vec{\xi}) \equiv (\epsilon_{11}, \epsilon_{12}^*, \epsilon_{21}, \epsilon_{22}^*)^T$ is the perturbation to the nonlinear eigenmode $(\bar{u}_1, \bar{u}_2)$ of Eqs. (5)–(7). By retaining in Eqs. (2) and (3) only linear terms in $\vec{\epsilon}$, we end up with the linear problem

$$\frac{d\vec{\epsilon}}{d\xi} = i \begin{pmatrix} -\Omega_1^2 - \bar{\Omega} & \bar{u}_2 & \bar{u}_1 & 0 \\ -\bar{u}_2 & \Omega_1^2 - \bar{\Omega} & 0 & -\bar{u}_1 \\ \bar{u}_1 & 0 & \bar{\Omega} - \Omega_2^2 & 0 \\ 0 & -\bar{u}_1 & 0 & \bar{\Omega} + \Omega_2^2 \end{pmatrix} \vec{\epsilon}, \quad (13)$$

where $\Omega_1^2 \equiv \Omega^2/2 + \theta$, $\Omega_2^2 \equiv \Omega^2/2 + \eta$, $\bar{\Omega} \equiv \delta\Omega/2$, and we set $\beta_1 = \beta_2 = 1$, $2\theta - \delta k = \eta = \pm 1$ in order to be consistent with the transformed variables in Eq. (7). The unstable frequencies Ω are those yielding at least one eigenvalue of the imaginary matrix in Eq. (13) with the positive real part. Let us consider, for the sake of simplicity, stationary waves such that $\delta = v = \nu = 0$ in Eq. (4). Equation (13) yields the potentially unstable eigenvalues

$$\lambda^{\pm} = \frac{\sqrt{f_0(\Omega) \pm \sqrt{f_0(\Omega)^2 - f_1(\Omega)}}}{2}, \quad (14)$$

where $f_0(\Omega) = -2(1+4\eta\theta) - 2(\eta+\theta)\Omega^2 - \Omega^4$ and $f_1(\Omega) = \Omega^2[\Omega^6 + 4(\theta+\eta)\Omega^4 + 4\Omega^2 - 16\theta(1+2\theta\eta)]$. By imposing $f_1 = 0$ in Eq. (14) we obtain the boundary of instability. MI occurs for $\Omega_d^2 < \Omega^2 < \Omega_u^2$ when $\eta = -1$ and for $0 < \Omega^2 < \Omega_d^2$ when $\eta = 1$, where $\Omega_d^2 = -\eta + \sqrt{1+8\eta\theta}$ and $\Omega_u^2 = 2(1-2\theta)$. In this case λ^+ is real, and the maximum MI gain $g = \lambda^+$ is shown in Fig. 2 as a function of Ω and θ, for the regime (i.e., $\eta = -1$) which allows for the propagation of kink and dark ($\theta = -1$) waves.

This MI is the result of combined TWM processes due to the interplay of dispersion and quadratic nonlinearity. Hence we expect that it may be relevant whenever *quadratic* nonlinearities play a significant role. This may occur, e.g., in the TWM of surface acoustic waves [20], for resonant (i.e., complex nonlinearity) interactions in water waves [21], as a mechanism of instability in plasmas [22], in optomechanical wave interactions which may constitute a seed for MI of the optical waves [23], and in driven-damped dissipative systems such as an optical cavity. Here we specifically show that MI affects the

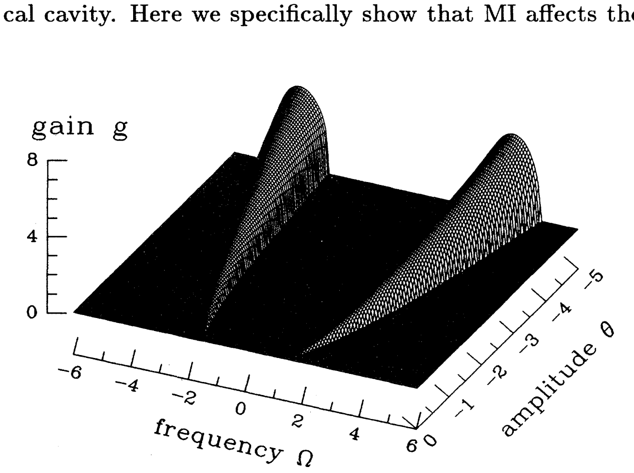

FIG. 2. Modulational instability gain vs frequency detuning Ω, and the amplitude dependent phase shift θ.

propagation of dark and kink waves. In Fig. 3 we show the spatiotemporal evolution obtained numerically from Eqs. (2) and (3), for the kink pair with $\theta = -0.311$ [see Fig. 2(d)]. Similar results hold for dark pairs. As shown, after the initial stage of unperturbed propagation, the cw backgrounds which sustain the localized wave forms start to break up spontaneously (i.e., the perturbation was not even seeded) into periodical patterns at the mostly unstable frequency (i.e., $\Omega = 1.75$ which yields $dg/d\Omega = 0$). The direction of the unstable eigenvector determines the phase shift between the two patterns. We also observed that MI does not exhibit any recursive behavior past the linearized stage of propagation, in contrast to MI of the integrable cubic NLSE. On a large scale length, we observed a transition to spatiotemporal turbulence involving the disordered generation of new frequencies. Our simulations also suggest that when the cw backgrounds are replaced by sufficiently long and smooth background pulses, MI is no longer the prevailing instability mechanism. The kink structures propagate unchanged for several dispersion lengths until they are destroyed by the dispersive broadening of the background tails.

Finally, we discuss the meaning of the dimensionless parameters in terms of physical units. The total background intensity of the eigenmodes $(\overline{u}_1, \overline{u}_2)$ is

$$I_t = |A_1|^2 + |A_2|^2 = (\theta_1^2 + \theta_1\theta_2)(\chi z_d)^{-2} = \delta k^2(3 - 2\theta + \beta_2)/[(2\theta - \beta_2)(\chi z_d)]^2,$$

where $\chi = [2\omega_0/(c^3\epsilon_0 n_{\omega_0}^2 n_{2\omega_0})]^{1/2} d^{(2)}$, and $d^{(2)}$ is the effective susceptibility element, whereas the fraction of second-harmonic intensity is $\rho = |A_2|^2/I_t = (3 - 2\theta + \beta_2)^{-1}$. For typical dispersions $\beta_1 \simeq \beta_2 = 1$ ps^2/m (we consider for simplicity $\delta = v = \nu = 0$), and a time scale $t_0 = 50$ fsec (significant field variations occur in Fig. 1 over $t = \tau t_0 \simeq 500$ fsec), the dispersion length is $z_d = 2.5$ mm. At $\lambda_0 = 2\pi c/\omega_0 = 1.06$ μm, for a mismatch $\delta k = \Delta k z_d \simeq 1$, and $d^{(2)}/\sqrt{n_{\omega_0}^2 n_{2\omega_0}} = 1$ pm/V, the condition $\theta = -0.566$ (see Fig. 1) yields a background intensity of $I_t \simeq 600$ MW/cm^2 and $\rho \simeq 0.2$. The mostly unstable angular frequency $\Omega = 1.9$ yields a modulation at frequency $f = \Omega/(2\pi t_0) \simeq 6$ Thz.

In conclusion, we have shown that the parametric two-wave interaction in a dispersive transparent medium with quadratic nonlinearity possesses periodical waves and exotic wave forms (invariant domain walls between the nonlinear eigenmodes of the process) which travel with a locked group velocity. A stability analysis against periodic perturbations suggest that the Benjamin-Feir instability, which is characteristic of dispersive *cubic* media, is a rather ubiquitous mechanism which occurs also in media with *quadratic* nonlinearity.

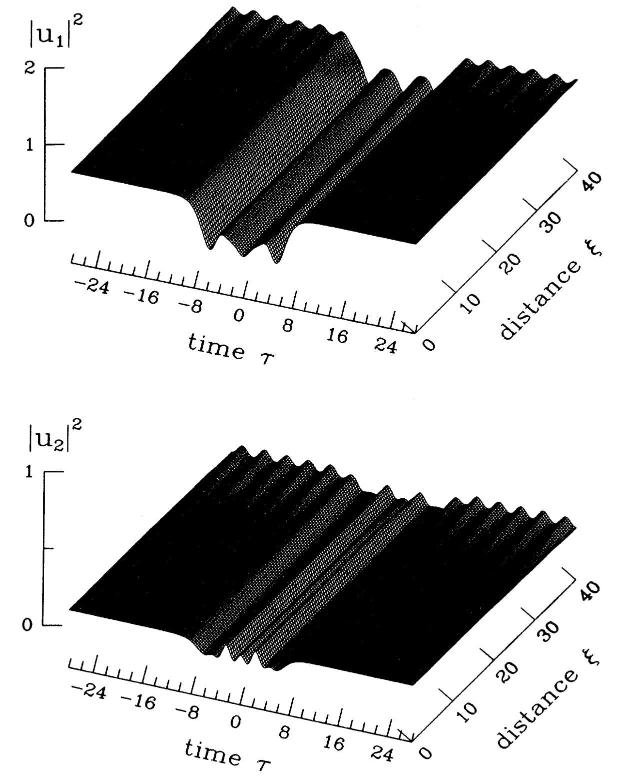

FIG. 3. Propagation of the localized domain wall solution for $\theta = -0.311$.

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